RC 13013 - REVISED 3/30/88 (#61018)
Computer Science 24 pages

Research Report

Connectivity Properties of a Packet Radio Network Model

T. K. Philips

IBM Research Division
T. J. Watson Research Center
Yorktown Heights, N.Y. 10598

S. S. Panwar

Department of Electrical Engineering and Computer Science
Polytechnic University
333 Jay Street
Brooklyn, N.Y. 11201

A. N. Tantawi

IBM Research Division
T. J. Watson Research Center
Yorktown Heights, N.Y. 10598

IBM Research Division
Almaden • Yorktown • Zurich

Copies may be requested from:

IBM Thomas J. Watson Research Center
Distribution Services F-11 Stormytown
Post Office Box 218
Yorktown Heights, New York 10598

CONNECTIVITY PROPERTIES OF A PACKET RADIO NETWORK MODEL

T. K. Philips[1], S. S. Panwar[2], and A. N. Tantawi[1]

[1]IBM T.J. Watson Research Center, Yorktown Heights, NY 10598

[2]Department of Electrical Engineering and Computer Science,
Polytechnic University, 333 Jay Street, Brooklyn NY 11201

ABSTRACT

A model of a packet radio network in which transmitters with a transmission range of R units are distributed according to a two dimensional Poisson point process is examined. It is a widely held belief that an optimal expected number of nearest neighbors of a transmitter (or equivalently an optimal transmission range) that maximizes the throughput of the network exists. This so called "magic number" is usually taken to be 6. In this paper we show that the number of nearest neighbors must grow logarithmically with the area of the network so as to ensure that the network is connected, implying that no magic number can exist. The notion of a magic number is still useful though, and an explanation is provided for why computations based on magic numbers give answers that are good in practice. This follows from a re-examination of a problem first examined by Gilbert [1]. He assumed the Poisson point process to generate points over the entire XY plane and showed that if the expected number of nearest neighbors of a point exceeded some critical value N_0 the plane would contain an infinite connected component. We show that $2.195 < N_0 < 10.526$.

This work was partially supported by the Center for Advanced Technology in Telecommunications, Polytechnic University, Brooklyn, NY 11201.

1. Introduction

A simplistic, though widely used model of a mobile packet radio network is based on the following assumptions.

1. Nodes in the network lie in a bounded figure of area A. We shall assume the figure to be a square.
2. The Homogeneous Poisson Assumption : An elemental area ds contains at most one node, the probability of this nodes existence being Dds, where D is the density of nodes in the plane.
3. Each node is able to communicate with any other node that is at most R units distant from it. The set of all such nodes is referred to as the nearest neighbors of the node.
4. $\pi R^2 << A$
5. The figure should not be "narrow", i.e, its width at the narrowest part should be much greater than R. Equivalently, edge effects should be negligible. This is true for a square.
6. All nodes generate Poisson streams of traffic at an identical rate.

Under these assumptions, with the further restriction that the medium access protocol be slotted Aloha, a number of authors [3, 5, 10] have shown that in order for the throughput to be maximized, we must have $\pi R^2 D \sim 6$, leading to the widely held belief that 6 is a "magic number". In this paper it will be shown that if $\pi R^2 D$ is a fixed constant, then for sufficiently large A the network will almost surely be disconnected, implying that no magic number can exist. This, however, does not render the notion of a magic number useless. If A is sufficiently large, it may be well approximated by the infinite XY plane. Gilbert [1] has shown that there is a critical number N_0 such that if $\pi R^2 D > N_0$, there is a non-zero probability that the random plane network contains an infinite connected component. He found N_0 to be bounded between 1.64 and 17.9. More recently, Hall [2] has shown that $2.186 < N_0 < 10.588$. We shall tighten these bounds to $2.195 < N_0 < 10.526$. Returning to the finite area A, we would expect the vast majority of nodes to be contained in a single giant component if $\pi R^2 D > N_0$. Gilbert ran simulations to verify this, and found it to be true in practice. In addition, he estimated the true value of N_0 to be about 3.2. It is conjectured that the true value is π. An attempt was made by Koch [6] to prove this conjecture, but the proof is incorrect.

The paper is organized as follows. In section 2, we determine the necessary and sufficient conditions for the plane to be covered (i.e. for every point in the square to lie at a distance of R or less from some Poisson point.) In section 3, these results are used to determine the necessary and sufficient conditions for the Poisson points to be connected. In section 4, Gilbert's problem [1] is re-examined and new bounds are found on the density of points required for an infinite connected component to exist.

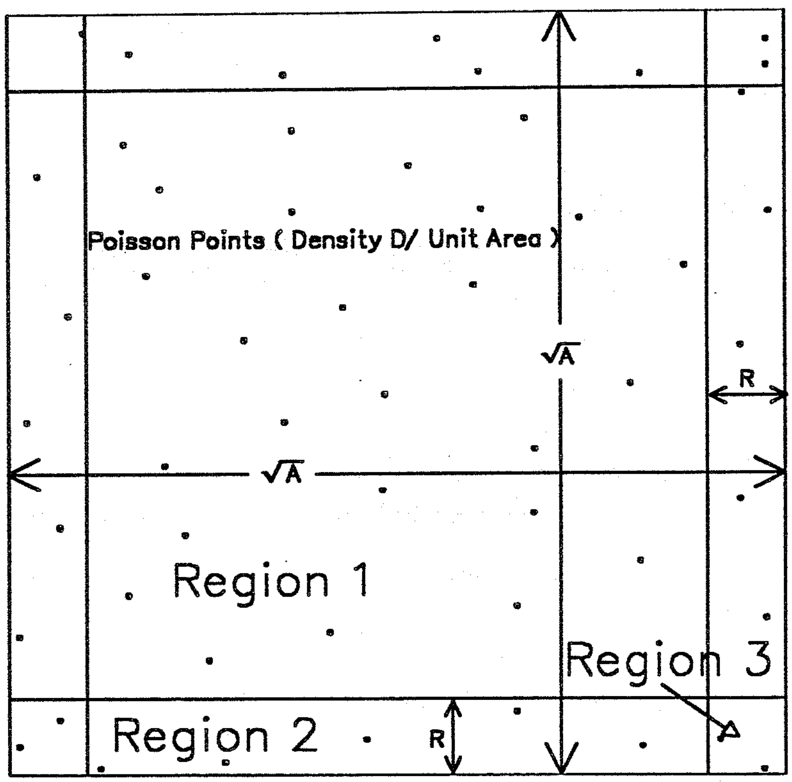

Figure 1. The Model.

2. Covering the Plane

First, we digress a little and consider a closely related problem - that of covering an area with randomly located circles. Consider a square of area A in which points are generated by a two dimensional Poisson point process of density D points per unit area. Each Poisson point is assumed to cover all points that lie within a radius R of it. The question posed is: given a functional form for R that may depend on D and A find $\lim_{A\to\infty}\Pr[\text{Square is covered}]$.

If we consider these Poisson points to be base stations or repeaters in a packet radio network, the circle of radius R would define the transmitting or receiving range of the station. Clearly, if the square is covered, every point in it will have access to at least one station.

The answer to this question is to be found in the following propositions which are closely related to, although somewhat weaker than, a very general theorem due to Miles [7]. The proof of these weakened versions is very much simpler than that of Miles' theorem. The key to the proofs is a not very well known theorem due to Robbins [9].

Theorem 0 :

Let S be a random Lebesgue measurable subset of R_n with measure $\mu(S)$. For any point $x \in R_n$, let $p(x) = \Pr[x \in S]$. Define

$$g(x,S) = \begin{cases} 1 \text{ if } x \in S, \\ 0 \text{ otherwise.} \end{cases}$$

Then assuming that the function $g(x,S)$ is a measurable function of the pair (x,S), the expectation of the measure of S is given by the Lebesgue integral of the function $p(x)$ over R_n.

Theorem 1:

For any $\varepsilon > 0$, if $R = \sqrt{\dfrac{(1-\varepsilon)\ln A}{\pi D}}$, then $\lim_{A\to\infty}\Pr[\text{Square covered}] = 0$.

Proof:

On the square of area A, construct a square lattice of side $2R$ containing $\lfloor \frac{\sqrt{A}}{2R} \rfloor^2$ points as shown in figure 2. The lattice is drawn so that it is centered over region 1. Then, if $\sqrt{A}$ is a multiple of $2R$, its outermost rows will lie on the boundary between region 1 and region 2, and if not, the lattice points will be contained entirely within region 1. If the plane is to be covered, every lattice point must be covered. For a lattice point to be covered, a Poisson point must lie within a circle of radius R centered at the lattice point. Therefore, in region 1 (refer to figure 1) we can write

$$\Pr[\text{ A specified lattice point is not covered }] = e^{-\pi R^2 D}.$$

Let Y be a random variable that counts the number of uncovered lattice points. Then we have

$$\mathrm{E}[Y] = \lfloor \frac{\sqrt{A}}{2R} \rfloor^2 \times e^{-\pi R^2 D} \tag{1}$$

and

$$\mathrm{E}[Y^2] = \mathrm{E}[Y] + 2 \times \binom{\lfloor \frac{\sqrt{A}}{2R} \rfloor^2}{2} \times e^{-2\pi R^2 D}. \tag{2}$$

Substituting $R = \sqrt{\frac{(1-\varepsilon)\ln A}{\pi D}}$, in (1) and (2) yields

$$\mathrm{E}[Y] = (1 + o(1))\frac{D\pi A^{\varepsilon}}{4 \ln A} \tag{3}$$

and

$$\mathrm{E}[Y^2] = \mathrm{E}[Y] + \frac{D^2\pi^2}{16(lnA)^2} \times A^{2\varepsilon} + o(A^{-.5}), \tag{4}$$

respectively.

Using a variation of Chebyshev's inequality [8] we get

$$\begin{aligned} \Pr[Y = 0] &\leq \frac{\mathrm{E}[Y^2] - \mathrm{E}^2[Y]}{\mathrm{E}^2[Y]} \\ &\leq \frac{1}{\mathrm{E}[Y]} + o(A^{-.5}) \\ &= o(1). \end{aligned} \tag{5}$$

It follows that in the limit as $A \to \infty$, there is almost surely an uncovered lattice point, and consequently the square is not covered.

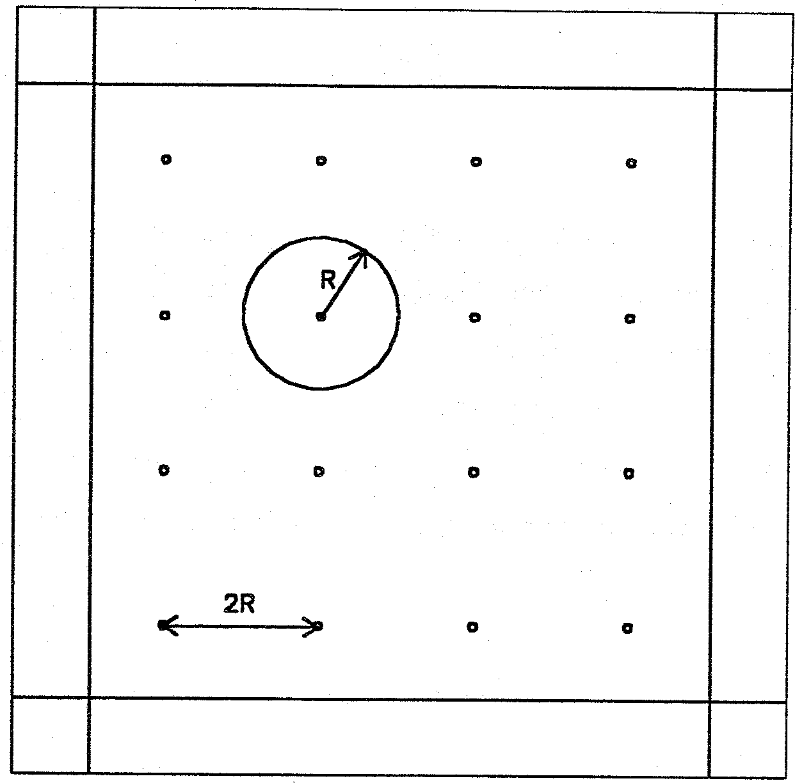

Figure 2. Covering the Plane: The Square Grid.

Theorem 2:

For any $\varepsilon > 0$, if $R = \sqrt{\frac{(1+\varepsilon)\ln A}{\pi D}}$, then $\lim_{A\to\infty} \Pr[\text{Square covered}] = 1$.

Proof:

Let $0 < \varepsilon < .25$. Using Robbins theorem, the expectation of the uncovered area is given by

$$E[\text{Uncovered Area}] = \int_0^{\sqrt{A}}\int_0^{\sqrt{A}} \Pr[\text{The point (x,y) is not covered}]dxdy$$
$$= \iint_{\text{Region 1}} + \iint_{\text{Region 2}} + \iint_{\text{Region 3}} \Pr[\text{The point (x,y) is not covered}]dxdy.$$

If $R = \sqrt{\frac{(1+\varepsilon)\ln A}{\pi D}}$, $\varepsilon > 0$, the first and last integral can be bounded, giving the expectation of the covered area to be

$$O(A^{-\varepsilon}) + O(\ln A \times e^{\frac{(1+\varepsilon)\ln A}{4}}) + 4\int_R^{\sqrt{A}-R}\int_0^R \exp(-D\left[\frac{\pi R^2}{2} - R^2 \sin^{-1}\frac{y}{R} - y\sqrt{R^2-y^2}\right])dydx$$
$$= O(A^{-\varepsilon}) + O(\ln A \times A^{-.25}) + 4\int_R^{\sqrt{A}-R}\int_0^R \exp(-D\left[\frac{\pi R^2}{2} - R^2 \sin^{-1}\frac{y}{R} - y\sqrt{R^2-y^2}\right])dydx. \quad (6)$$

The integral over region 1 is obvious. The integral over region 3 is is upper bounded by observing that the square must intersect at least one fourth of the circle surrounding a Poisson point, and that the area in the region is proportional to $\ln A$. This leaves us only the integral over region 2 to evaluate. The exponent of e in the integral is the negative of the area contained in the intersection of a square of area $A > R^2$ and a circle of radius R whose center lies at a distance of y from the boundary. The integral can be upper bounded by lower bounding the terms in the exponent. Using

$$R^2 \sin^{-1}\frac{y}{R} \geq yR$$

and

$$y\sqrt{R^2-y^2} \geq 0,$$

we get

$$\begin{aligned} \text{E[Uncovered Area]} &= O(A^{-\frac{\varepsilon}{2}}) + O(A^{-\varepsilon}) + O(\ln A \times A^{-.25}) \\ &= O(A^{-\frac{\varepsilon}{2}}) \end{aligned} \quad (7)$$

We also need a lower bound on E[Uncovered Area | Uncovered Area > 0]. If the square is not covered, then there must be at least one point in the square that is at a distance of R or greater from any Poisson point as shown in figure 3. We may draw a circle of radius R around this point that is devoid of Poisson points. Outside this circle, the Poisson point process generates points at a density of D points/unit area. Clearly we have

$$\text{E[Uncovered Area | Uncovered Area} > 0] > \text{E[Area in this circle that is uncovered]}. \quad (8)$$

In Appendix 1 the expected area in the circle that is uncovered is shown to be $\Omega(\frac{1}{\ln A})$. Now

$$\begin{aligned}\Pr[\text{ Square covered }] &= 1 - \frac{\text{E}[\text{Uncovered Area }]}{\text{E}[\text{Uncovered Area } | \text{ Uncovered Area } > 0]} \\ &> 1 - \frac{\text{E}[\text{Uncovered Area }]}{\text{E}[\text{Uncovered Area in circle}]} \\ &= 1 - O(\ln A) \times O(A^{-\frac{\varepsilon}{2}}) \\ &= 1 - o(1).\end{aligned} \tag{9}$$

Lastly note that the probability that the square is covered is a non decreasing function of ε, and consequently the results hold for all $\varepsilon > 0$.

■

Observation:

The average number of nearest neighbors needed to guarantee that the area is covered is given by $\pi\left[\frac{(1+\varepsilon)lnA}{\pi D}\right]D$ or a little more than $\ln A$.

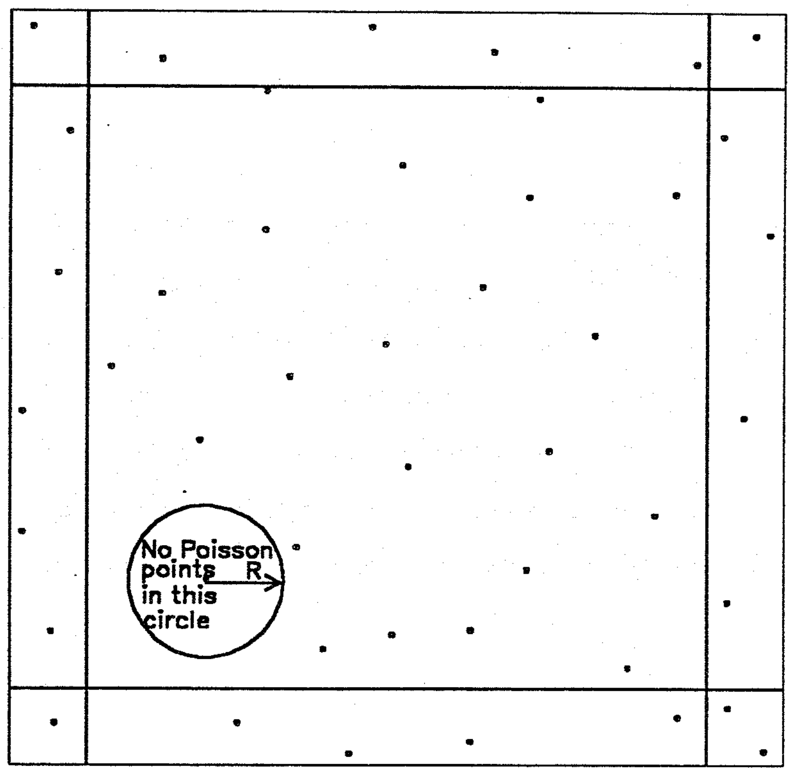

Figure 3. Covering the Plane: The Depopulated Region.

3. Connectivity

Once again consider our square of area A in which points or repeaters are generated by a two dimensional Poisson point process of density D. Now, each node is assumed to be connected to all nodes that lie within a radius R of it. From the view point of a packet radio network, if the network is connected any Poisson node can route a message to any other Poisson node. If we think of the Poisson nodes as repeaters, then if the network is connected, a transmitter that lies within the range of any one repeater can communicate with a receiver located within the range of any other repeater, and this points to a simple way to ensure radio coverage of a region. If, on the other hand, the Poisson points are thought of as transceivers, as in a mobile radio network, the connectedness of the network allows communication between any pair of transceivers. Once again, given a functional form for R that may depend on D and A, we investigate the behavior of $\lim_{A\to\infty} \Pr[\text{Network is connected}]$.

Theorems 1 and 2 give us a threshold for coverage. If $\pi R^2 D$, the average number of nearest neighbors of a randomly chosen node is "slightly less" than $\ln A$, the square is almost surely not covered, while if it is "slightly greater" than $\ln A$, it is almost surely covered. Though connectivity and coverage are independent - neither implies the other - intuitively one would expect the two properties to have very similar thresholds. A simple argument to justify this assertion goes as follows. Suppose first that $R = \sqrt{\frac{(1-\varepsilon)\ln A}{\pi D}}$. Then the expectation of the uncovered area is $O(A^{\varepsilon})$ which goes to ∞ as $A \to \infty$, and one might reasonably expect that for A sufficiently large, an uncovered patch would exist between two components in the graph. Likewise if $R = \sqrt{\frac{(1+\varepsilon)\ln A}{\pi D}}$, the square is almost surely covered, implying that the Poisson points lie very "close" to each other. We might therefore reasonably expect the probability of the network being connected to be close to 1. This heuristic argument is put on a firmer footing in the next two propositions.

Theorem 3:

For any $\varepsilon > 0$, if $R = \sqrt{\frac{(1-\varepsilon)\ln A}{\pi D}}$, then $\lim_{A\to\infty} \Pr[\text{Network is connected}] = 0$

Proof:

Referring to figure 1 , if a single node that lies in region 1 is isolated, the network is not connected. To show that an isolated node exists with high probability we proceed as follows.

1. Find the first two moments of the number of isolated nodes.

2. Use the second moment method to show that the probability of finding an isolated node goes to 1 as $A \to \infty$.

Let X be a random variable that counts the number of isolated nodes in region 1. Then

$$\Pr[\text{ a specified node is isolated }] = e^{-\pi R^2 D}$$

and

$$E[X] = (\sqrt{A} - 2R)^2 D e^{-\pi R^2 D}. \tag{10}$$

If $R = \sqrt{\frac{(1-\varepsilon)\ln A}{\pi D}}$, we have

$$\begin{aligned} E[X] &= (\sqrt{A} - 2R)^2 D e^{-(1-\varepsilon)\ln A} \\ &= (1 - o(1)) D A^{\varepsilon}. \end{aligned} \tag{11}$$

The expected number of isolated nodes grows without bound as Λ is increased. It remains only to show that the probability of finding one or more isolated nodes approaches 1 as $A \to \infty$. This is most easily accomplished by the use of Chebyshev's inequality. To find the second moment, define indicator random variables $\{x_i\}$, $x \geq 1$, such that $x_i = 1$ if the i^{th} node is isolated, and 0 if it is not. Then $X = \sum_i x_i$, and

$$\begin{aligned} E[X^2] &= E\left[\sum_i x_i^2 + \sum_{i \neq j} x_i x_j\right] \\ &= E[X] + (1 + o(1)) E^2[X], \end{aligned} \tag{12}$$

provided that $\pi R^2 D > (1 + \varepsilon) \ln A$.

In the last step we use the fact that $x_i = x_i^2$ to conclude that the first term is E[X]. To see that the second term is $(1 + o(1))\mathrm{E}^2[X]$, first evaluate it conditioned on the existence of exactly m points and then uncondition it using the fact that the number of nodes in region 1 is a Poisson random variable. Finally use the condition on $\pi R^2 D$ to conclude that the subdominant terms are $o(1)$ with respect to the dominant term.

Using a variation of Chebyshev's inequality [8] we get

$$\Pr[X=0] \le \frac{\mathrm{E}[X^2] - \mathrm{E}^2[X]}{\mathrm{E}^2[X]}$$
$$= \frac{1}{\mathrm{E}[X]} + o(1).$$

If $R = \sqrt{\frac{(1-\varepsilon)\ln A}{\pi D}}$, then we have

$$\Pr[X=0] = O(A^{-\varepsilon}) + o(1)$$

or

$$\Pr[X>0] = 1 - o(1). \quad (13)$$

It follows that the probability of the graph being connected also goes to 0 as $A \to \infty$.

■

Proving the contrapositive of this last statement is extremely difficult. Clearly, if $R = \sqrt{\frac{2 \times (1+\varepsilon)\ln A}{\pi D}}$, $\varepsilon > 0$, the graph is almost surely connected. This follows from the observation that if $R = \sqrt{\frac{(1+\varepsilon)\ln A}{\pi D}}$ the square is almost surely covered, implying that any two Poisson point are separated by no more than 2R. If the "radius of influence" is now doubled, points that lie in different components will be connected, resulting in a connected graph. If we make two simplifying assumptions, which though almost surely true, are very hard to prove, a strong contrapositive to the last statement can be proven. The assumptions are:

1. The addition of a single node to the plane at a random point does not change the asymptotic probability of the graph being connected.

2. If the graph is disconnected, it almost surely contains only two components.

Theorem 4:

Under the above two assumptions, for any $\varepsilon > 0$, if $R = \sqrt{\frac{(1+\varepsilon)\ln A}{\pi D}}$, then $\lim_{A\to\infty} \Pr[\text{Network is connected}] = 1$.

Proof:

Denote by P' the probability that the graph is connected when a node is added, and by P the probability that it is connected without the addition of a node. Then for some constant c

$$P' > P \times (1 - O(A^{-(1+\frac{\varepsilon}{2})})) + (1-P) \times \frac{c}{A}(1 - o(1)). \tag{14}$$

The terms in the equation arise as follows. If the original graph is connected, the graph that results from the addition of a node will be connected if the added node falls into the portion of the square that is covered. But if $R = \sqrt{\frac{(1+\varepsilon)\ln A}{\pi D}}$ the uncovered area is $O(A^{-\frac{\varepsilon}{2}})$ and this accounts for the first term. If the graph is not connected, by assumption (2) it contains only two components, and if the added node lands in the intersection of the two components the resulting graph will be connected. If the square is covered, the expected area in the intersection is at least as large as some fixed constant c. This constant multiplied by $(1 - o(1))$, the probability of the square being covered, gives us a lower bound on the area in which the added node must fall if it is to connect the two components.

By assumption (1), $P' = P$, and consequently equation (14) reduces to

$$P > \frac{(1-o(1))c}{(1-o(1))c + O(A^{-\frac{\varepsilon}{2}})} \tag{15}$$

proving our assertion.

■

Corollary:

There can be no magic number, as the expected number of nearest neighbors needed to ensure connectivity grows logarithmically with the area of the square in which the points lie.

4. The Infinite Component

As indicated in the corollary to proposition 4, no magic number can exist. In this section we explore the properties of random plane networks of low density and use them to conclude that the concept of a magic number is useful in practice. Gilbert [1] showed that if the Poisson process was assumed to generate points over the entire XY plane and if the average number of nearest neighbors of a point exceeded some fixed constant N_0, an infinite component would exist with non-zero probability. He found $1.64 < N_0 < 17.9$. (Due to a typographical error, the lower bound on N_0 in [1] is given as 1.75.) More recently, Hall [2] has shown that $2.186 < N_0 < 10.588$. Note that the existence of an infinite component does *not* imply that all the nodes are connected. There will, by Theorem 3, be infinitely many isolated nodes. When the area of the square is finite, no infinite component can exist, though we might reasonably expect most of the nodes to belong to a giant component. Gilbert carried out simulations and found this to be true in practice. From the simulation, he concluded that the $N_0 \sim 3.2$. It has been conjectured that the true value of N_0 is π, though no proof of this has been found.

Consider a random plane network generated so that the average number of nearest neighbors of a randomly chosen node is 6. By Robbins theorem, the fraction of the square that is uncovered is approximately e^{-6}, and the fraction that is singly covered is about $6e^{-6}$. Together, they account for less than 1.75 % of the area of the square. It is reasonable to expect much of the singly covered area to be composed of isolated nodes and portions of the boundary of the giant component (assuming that $N_0 < 6$). If the portion of the singly connected area that consists of isolated nodes is erased, the remaining portion should not really differ appreciably from our model of a two dimensional Poisson point process of density D points/unit area. In addition, as the uncovered and singly covered area is a small fraction of the total area, the giant component would tend to cover most of the plane. These facts taken together explain the generally good agreement between theory and simulation reported in [3].

Finally we present two theorems that provide tighter bounds on N_0 than those found in [2].

Theorem 5:

$N_0 > 2.195$

Proof:

We relate this problem to the multiclass M/D/1 queueing system described and analyzed in Appendix 2. Then, by studying the saturated M/D/1 queueing system we establish that a lower bound on N_0 is 2.195.

Imagine a scanner which scans the plane in a manner which we describe below; it scans a unit area of the plane in a unit time. This scanner corresponds to the server in the queueing system and its detection of a point corresponds to the arrival of a customer. Thus, arrivals form a Poisson process with intensity D. Consider the following scenario. Assume that the scanner starts at an arbitrary Poisson point a. It starts by scanning the area inside the circle A, which is centered at point a and has a radius R. This is analogous to an idle server which begins servicing an arriving customer a. While scanning A, the scanner sees another Poisson point b. This corresponds to the arrival of a second customer b while the server is still busy with customer a. Assuming a non-preemptive scheduling discipline, customer b is queued until customer a is out of the server, i.e. area A is completely scanned. Let x be the distance between point a and point b. We call customer b a class-x customer. The set of customer classes C is the interval $[0,R]$. The scanner then starts scanning area B. As the portion of area B which overlaps A has already been scanned, the scanner moves on to the remaining portion of B which does not overlap A. This area depends only on x : it is denoted by $S(x)$ and corresponds to the service time of customer b which is of type x. The scanning process continues until the scanner runs out of points. The scanner then starts scanning the uncovered part of the plane till it sees another point, corresponding to an idle server, beginning a new busy period. Note that the length of the busy period corresponds to the area of a component in the plane. Thus, an infinite component maps into a saturated queueing system where the server is busy with probability one.

In order to account for the service times of a sequence of customers, a, b, c, ..., arriving in a busy period we need to consider the overlap of their corresponding circles. We simplify this problem by considering only the overlap of circles of subsequent pairs of customers, i.e. (a,b), (b,c), ... With this assumption, the scanner covers more area than it would have had in the actual system. Therefore, a minimum arrival rate D^* which results in a saturated server in this simplified system should be less than the critical saturation rate D_0 in the actual system. Hence, by analyzing our saturated queueing system, we obtain D^* which is a lower bound on D_0. It follows that $N^* = \pi r^2 D^*$, the average number of nearest neighbors of a point is a lower bound on N_0.

The service time of a class-x customer, $S(x)$, is the area of the lune formed by the intersection of two circles of radius R whose centers are separated by x. It can be shown that

$$S(x) = 2R^2 \sin^{-1}(\frac{x}{2R}) + x\sqrt{R^2 - \frac{x^2}{4}}, \tag{16}$$

$0 \le x \le R$. Consider the circle which contains the lune of area $S(x)$. Draw a concentric circle of radius y and let $l(y,x)$ be the length of the arc of this circle which lies inside the lune. We have

$$l(y,x) = 2y \cos^{-1}(\frac{R^2 - y^2 - x^2}{2yx}), \tag{17}$$

$R - x \le y \le R$ and $0 \le x \le R$. Substituting $S(x)$ and $l(x,y)$ from equations (16) and (17), respectively, into equation (24) in appendix 2 yields $p^*(x)$, $0 \le x \le R$, which is the density function of the class of a randomly chosen Poisson point. We then solve equation (24) numerically and substitute $p^*(x)$ into equation (23) to obtain D^*, the critical density of points for which an infinite component results. Finally, we obtain the critical expected number of nearest neighbors of a point, $N^* = \pi r^2 D^* = 2.195$.

■

Theorem 6:

$N_0 < 10.526$

Proof:

Construct a site percolation process [4] on a triangular lattice on the XY plane as shown in figure 4.

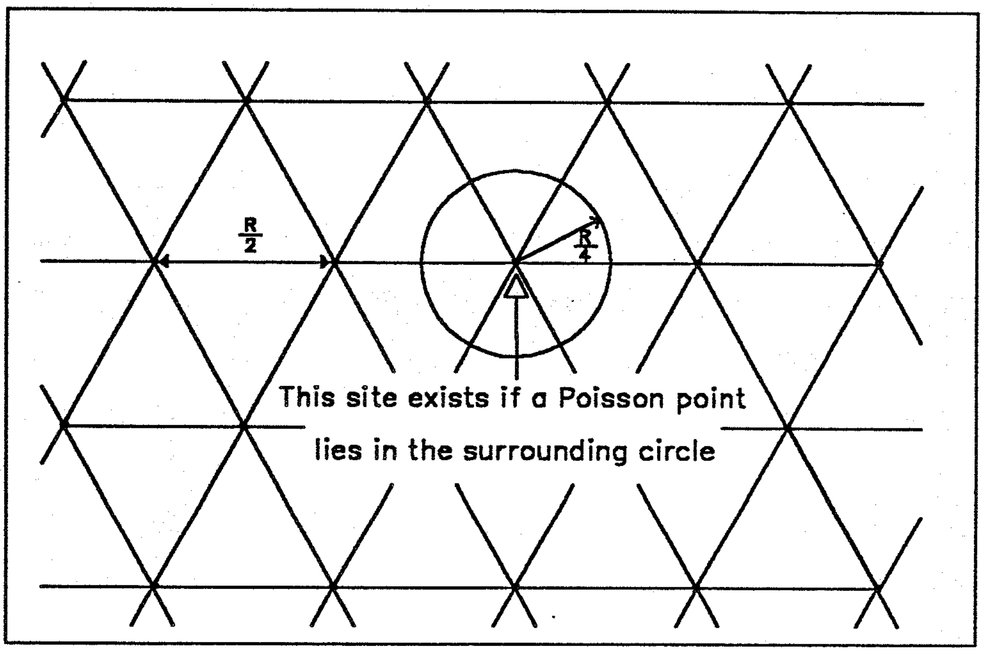

Figure 4. Site Percolation on a Triangular Lattice.

The distance between lattice points is $\frac{R}{2}$. Each lattice point exists independently of all other lattice points with probability p. If a lattice point does not exist, the edges joining it to its neighbors are also erased. Kesten [4] has shown that if $p > .5$ an infinite component exists with probability 1.

We shall consider a lattice point to exist if a Poisson point lies within a distance of $\frac{R}{4}$ of it. Note that if the spacing between lattice points is $\frac{R}{2}$, they exist independently of each other. Consider two lattice points that are joined by an edge. If both exist, then each one must have at least one Poisson point closer than $\frac{R}{4}$ to it. As the distance between these Poisson points is less than R, they are connected to each other. Clearly then, if an infinite component exists in the lattice, an infinite chain of connected Poisson points can be found. Therefore, if

$$1 - e^{-\pi(\frac{R}{4})^2 D} > .5, \tag{18}$$

an infinite component can be found in the Poisson process. This can be rearranged to give

$$\pi R^2 D > 16 ln 2 = 11.1. \tag{19}$$

But $\pi R^2 D$ is the expected number of nearest neighbors of a randomly chosen Poisson point and it follows that

$$N_0 < 11.1. \tag{20}$$

To tighten this to $N_0 < 10.526$, a simple modification is made to the preceding argument. Once again, we tesselate the plane with a triangular lattice, but a lattice point is now assumed to exist if a Poisson point is contained within a six sided figure that is constructed as follows. First, with a lattice point as center, construct a hexagon of side $\frac{R}{\sqrt{12+\alpha^2}}$ where α is a parameter, and $0 \le \alpha \le 1$. Draw circles of radius $\frac{R}{2}$ centered at the center of each of the six sides of the hexagon. The boundary of the figure is determined at each point by the boundary of the constituent figure (circle or hexagon) closest to the center of the hexagon. When $\alpha = 1$, the resulting figure is a hexagon (the circles just touch the extremities of the hexagon) and when $\alpha = 0$ the figure reduces to the "curved sided hexagon" used by Hall in [2]. Some elementary, though tedious, geometry shows that

1. The lattice points are separated by $R\sqrt{\frac{12}{12+\alpha^2}}$,
2. Points in adjacent figures are separated by at most R, and
3. The area of the figure attains its maximum of $.20688183R^2$ when $\alpha = .47178$.

Analyzing this system in the same way as we did the last one, we get get $N_0 < 10.525748$.

■

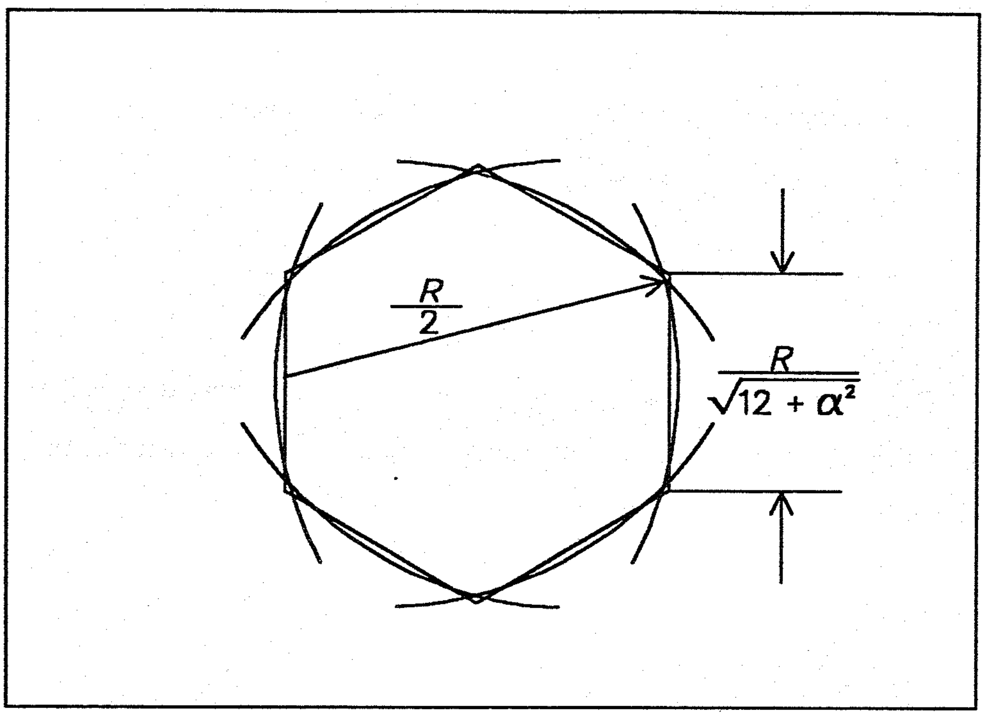

Figure 5. Site Percolation on a Triangular Lattice- the six sided figure.

5. Conclusions

A model of a packet radio network has been examined and it has been shown that no optimal number of nearest neighbors or "magic number" can exist. The notion of a magic number is shown to be useful though and an explanation for the generally good agreement between theory and simulation has been presented. A number of open questions remain. The single most important one concerns the homogeneity assumption made about the Poisson process. Extending these results to the inhomogeneous case will mark an important step forward. Another assumption that bears investigation is that of each node being connected to all nodes that lie within a circle of radius R around it. Certainly this is not valid in many environments of interest. These questions are significantly more difficult to resolve than those examined in this paper. We do, however, conjecture that the following hold.

Conjecture 1:

The conditions for connectivity and coverage are insensitive to the shape of the region that a Poisson point covers. This may not be unduly difficult for convex figures (such as the circle considered in this paper).

Conjecture 2:

If the Poisson process is non-homogeneous, then the "radius of influence" around each node that is needed to guarantee almost sure connectivity and almost sure coverage will be such that its expected number of nearest neighbors is $\geq (1 + \varepsilon) \ln A$.

References

1. Gilbert, E.N., "Random Plane Networks", J. S.I.A.M. Vol. 9, No. 4, 1961, pp 533-543.

2. Hall, P., "On Continuum Percolation", Annals of Probability, Vol. 13, No. 4, pp 1250-1266.

3. Hou, T.C. and Li, V.O.K, "Transmission Range Control in Multihop Packet Radio Networks", IEEE Trans. Commun., Vol. COM-34, No.1, Jan. 1986, pp 38-44.

4. Kesten, H., "Percolation Theory for Mathematicians", Birkhauser, Boston, 1982.

5. Kleinrock, L. and Silvester, J., "Optimum Transmission radii in Packet Radio Networks or why Six is a Magic Number", Proc. Nat. Telecommunications Conf., Birmingham, AL, Dec. 1978, pp 04.3.1-04.3.5.

6. Koch, D. W., "The Theory of Random Graphs with Applications to the Probabilistic Analysis of Optimization Algorithms", Ph.D. Thesis, UCLA, 1977.

7. Miles, R.E., "On the Homogeneous Planar Poisson Point Process", Math. Biosciences, Vol. 6, 1970, pp 85-127.

8. Palmer, E.M., "Graphical Evolution", John Wiley and Sons, New York, 1985, pg 138.

9. Robbins, H.E., "On the Measure of a Random Set", Ann. Math. Stat., Vol. 15, 1944, pp 70-74.

10. Takagi, H., and Kleinrock, L. "Optimal Transmission Ranges for Randomly Distributed Packet Radio Networks", IEEE Trans. Commun., Vol. COM-32, No. 3, Mar. 1984, pp 246-257.

Appendix 1

Consider a circle C of radius R in which no Poisson points are found. Outside the circle Poisson points are generated at a density of D points/unit area, as shown in figure 7. Assume the center of the circle to lie in region 1 of the square. Consider the point p at a distance of r from the center of the circle. Draw a circle C_1, also of radius R centered at p. If p is to be covered, the portion of C_1 that lies outside C must contain a Poisson point. The area of this lune shaped region is given by

$$2\left[R^2 \sin^{-1}\frac{r}{2R} + \frac{r}{4}\sqrt{4R^2 - r^2}\right] \triangleq L(r).$$

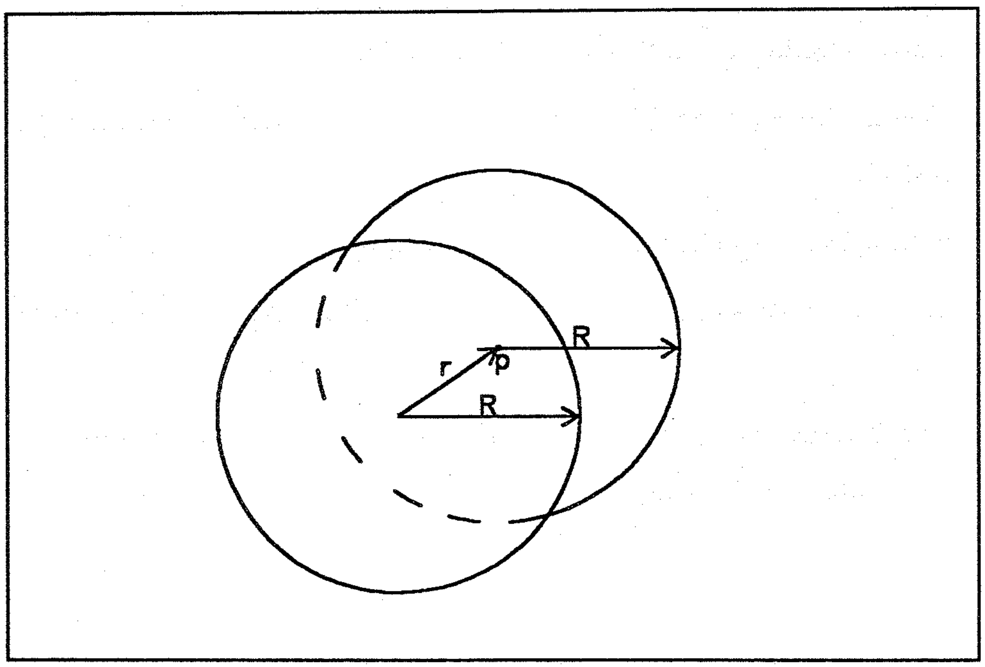

Figure 6. Lower Bounding the Uncovered Area.

The probability that p is uncovered is given by

$$e^{-DL(r)}$$

and by Robbins theorem, the expectation of the uncovered area is

$$\int_0^R 2\pi r e^{-DL(r)} dr.$$

The integral can be lower bounded by upper bounding $L(r)$. Now, $\sqrt{4R^2 - r^2} < 2R$ and from the convexity of the arcsine, $\sin^{-1}\frac{r}{2R} < \frac{\pi r}{6R}$ if $0 < r < R$ so that the expectation of the uncovered area is lower bounded by

$$\int_0^R 2\pi r e^{-2D\frac{\pi+1}{3}rR}dr.$$

This integral may be evaluated and shown to be $\Omega(\frac{1}{R^2})$, which is $\Omega(\frac{1}{\ln A})$ when $R = \sqrt{\frac{(1+\varepsilon)\ln A}{\pi D}}$.

Lastly, note that if the center of the circle were to lie in region 2 or region 3, the uncovered area could be reduced by at most a constant factor, and consequently we have the expectation of the uncovered area to be $\Omega(\frac{1}{\ln A})$ regardless of where the uncovered patch lies in the square.

Appendix 2

Consider a multiclass M/D/1 queueing system with a state-independent, time-invariant Poisson arrival process at rate λ. The customer classes are assumed to lie in some interval C on the real line. Note that there are uncountably many customer classes. Further, let $S(x)$ and $\lambda(x)$ denote the constant service requirement and the density of the arrival rate of class-x customers, respectively. It follows that the density of the utilization of the server due to class-x customers, which we denote by $\rho(x)$, is

$$\rho(x) = \lambda(x)\, S(x),$$

$x \in C$. By integrating the above equation, we write the server utilization, ρ, as

$$\rho = \int_{x \in C} \rho(x)\, dx.$$

The class of a customer is determined at arrival time, depending on the class of the customer present in the server. Let $\phi(x,y)$, $x,y \in C$, be the class transition probability density, i.e. the probability that a customer belongs to class x given that the customer in service at the time of arrival is of class y. An arriving customer to an empty system belongs to class x with probability $\phi_0(x)$, $x \in C$.

Given the service times of customers $S(x)$, $x \in C$, and the class transition probability densities $\phi(x,y)$ and $\phi_0(x)$, $x,y \in C$, we now obtain $\lambda(x)$ and $\rho(x)$, $x \in C$. Let the probability that an arrival is of class x be $p(x)$, $c \in C$, which is defined as $p(x) = \lambda(x)/\lambda$ where

$$\lambda = \int_{x \in C} \lambda(x)\, dx.$$

By conditioning on the class of the customer found in service at arrival time, we may write

$$p(x) = \int_{y \in C} \pi(y)\, \phi(x,y)\, dy + \pi_0\, \phi_0(x),$$

$x \in C$, where $\pi(y)$ is the probability that an arrival finds a class-y customer in service and π_0 is the probability that an arrival finds the system idle. Since the arrival process is Poisson, we have that $\pi(y) = \rho(y)$, $y \in C$, and $\pi_0 = 1 - \rho$. Thus, we obtain

$$p(x) = \int_{y \in C} \rho(y)\, \phi(x,y)\, dy + (1-\rho)\, \phi_0(x), \tag{21}$$

$x \in C$, or

$$p(x) = \lambda \int_{y \in C} p(y)\, S(y)\, \phi(x,y)\, dy + (1-\rho)\, \phi_0(x), \tag{22}$$

$x \in C$.

We now study the behavior of this queueing system as the server becomes saturated, i.e. $\rho \to 1$. Let λ^* and $p^*(x)$, $x \in C$, denote the total arrival rate and the class probabilities in a saturated system. In this case, equation (22) becomes

$$p^*(x) = \lambda^* \int_{y \in C} p^*(y)\, S(y)\, \phi(x,y)\, dy$$

$x \in C$.

Define $l(x,y)\, dx$ to be the interval of time during which customers of classes in the interval $(x, x + dx)$ may arrive during the service of a class y customer, i.e.

$$l(x,y) = S(y)\, \phi(x,y),$$

$x,\ y \in C$. Noting that

$$\lambda^* = 1 / \int_{y \in C} p^*(y)\, S(y)\, dy, \tag{23}$$

we write the above equation as

$$p^*(x) = \frac{\int_{y \in C} p^*(y)\, l(x,y)\, dy}{\int_{y \in C} p^*(y)\, S(y)\, dy}, \tag{24}$$

$x \in C$. Equation (24) satisfies the condition $\int_{x \in C} p^*(x)\, dx = 1$.