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# Research Report

## Approximating the Mean Waiting Time under the Golden Ratio Policy

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## ABSTRACT

We examine the Golden Ratio Policy of Hofri and Rosberg [1]. A simple approximate expression is developed for the mean waiting time of a packet. The computational requirements are minimal.

# I. INTRODUCTION

Hofri and Rosberg [1] analyze a TDM control policy called the Golden Ratio Policy for the Multiple Access channel in which capacity is allocated as follows: suppose there are $S$ transmitters, and let their requirements be $X_1, \ldots, X_S$ slots respectively, and let $\sum_{i=1}^{S} X_i \triangleq N$. Let $\phi^{-1} \triangleq \frac{\sqrt{5}-1}{2} = .6180339.....$, and let $x \bmod 1 \triangleq x - \lfloor x \rfloor$. Mark off the points $\phi^{-1} \bmod 1$, $2\phi^{-1} \bmod 1$, ..., $N\phi^{-1} \bmod 1$ on a circle of circumference 1, dividing it into N intervals. Allocate to session 1 the slots starting at $\phi^{-1} \bmod 1$, ..., $X_1\phi^{-1} \bmod 1$, to session 2 those starting at $(X_1+1)\phi^{-1} \bmod 1$, ..., $(X_1+X_2)\phi^{-1} \bmod 1$, and so on till all the slots have been allocated. Lastly, equalize the slot lengths. In effect, a sufficiently long frame length is chosen , and slots allocated within the frame in a reasonably regular manner. $\phi$ is also known as the Golden Ratio, explaining the appearance of the term in the title. The arrival process is characterized by the generating function of the number of arrivals in a slot and its first two moments.

Under the above policy, expressions for the mean and the distribution of the queue length at the beginning of a random slot are derived in [1]. The expressions, however, are complex. They require substantial amounts of computation to evaluate, and are consequently not very useful for quick "back of the envelope" calculations.

In the course of an investigation into the suitability of the Golden Ratio policy for flow control, the need arose for a simple expression for the mean waiting time of a packet when the arrival process was Poisson. In this note we present an approximation for the mean waiting time that is accurate to within a few percent.

The paper is organized as follows. First, the results of a simulation study of the Golden Ratio policy are presented. Next, an approximation for the mean waiting time is developed. Finally, the two are compared.

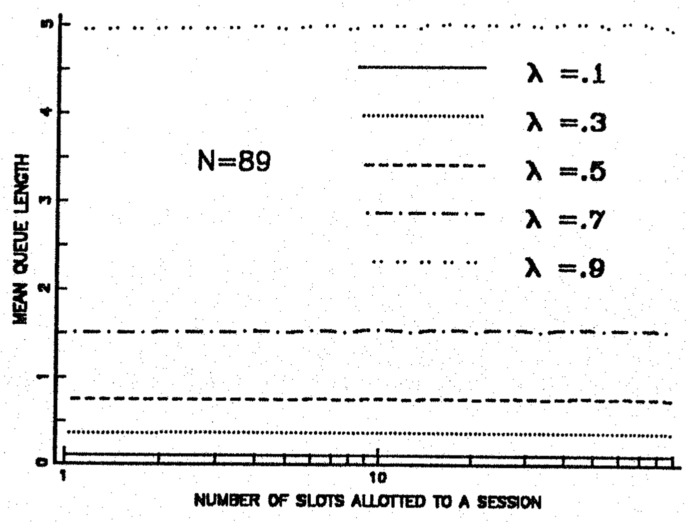


Figure 2. Mean queue length at the start of a slot.

Notice that the mean queue length and the mean waiting time depend strongly on $\lambda$, but only weakly on $X_1$. This observation is the key to our approximation. The mean queue length is solved for exactly when $X_1 = N$, and this is then used to approximate the mean queue length for all other values of $X_1$.

Consider a Golden Ratio policy and suppose that

1. There are a total of $N$ available slots, all of which are allotted to a single session,
2. The length of the cycle is $T$ seconds, and
3. The arrival process is Poisson, and the mean number of arrivals in a slot is $\lambda$ ($\lambda < 1$).

Then, from [2], the following hold.

1. The queue length at the start of a random slot has a steady state distribution $\pi = \pi_0,\ \pi_1, \ldots$
2. $\pi_0 = 1 - \lambda$
3. The mean queue length , $\overline{X}$,is given by $\dfrac{\lambda[2-\lambda]}{2[1-\lambda]}$.

The waiting time for a random arrival can now be written as mented (decremented) by the excess (shortfall) in the mean number of arrivals in the previous interval. Define $\overline{W}(X_1,N)$ to be the mean delay when $X_1$ out of $N$ slots are allotted to session 1. Then

$$\overline{W}(X_1, N) \approx \left[ \sum_{i=1}^{X_1} \frac{d_i^2(1+\lambda)}{2N\bar{d}} + (1-\pi_0)\left\{ \frac{1}{1-\pi_0} \sum_{i=1}^{X_1} \frac{d_i}{N} \left(\overline{X} + \frac{(d_{i-1}-\bar{d})\lambda}{\bar{d}}\right) - 1 \right\} \right] \frac{T}{X_1}$$
$$= \left[ (1+\lambda)\sum_{i=1}^{X_1} \frac{d_i^2}{2N\bar{d}} + \frac{\lambda^2}{2[1-\lambda]} + \lambda \sum_{i=1}^{X_1} \frac{d_i(d_{i-1}-\bar{d})}{N\bar{d}} \right] \frac{T}{X_1}. \tag{3}$$

The expression derived here was compared to the simulation, and the error in the approximation is plotted for in Figure 3 for various values of $\lambda$. From the figure we see that our approximation tends to over estimate the mean waiting time. This is because packets tend to arrive in long interpermit intervals and to be served in short ones, especially at intermediate utilizations. The variation in interpermit distances is minimized when $X_1$ is a Fibonacci number (see Theorem 5.1 of [1] ) and in these cases the approximation is seen to more accurate. The expression is exact when $\lambda = 0$. In any event, the error (when compared to the sample mean) is at most 2.5 percent, and this, taken in conjunction with the earlier mentioned confidence intervals, implies that with probability .997, the error is at most 3.5 percent. This is sufficient for most practical purposes. The expression can be simplified if desired by dropping the last term to give

$$\overline{W}(X_1, N) \approx \left[ (1+\lambda)\sum_{i=1}^{X_1} \frac{d_i^2}{2N\bar{d}} + \frac{\lambda^2}{2[1-\lambda]} \right] \frac{T}{X_1}, \tag{4}$$

in which case the maximum error rises to just under 3% (or 4% with probability .997). On the other hand, the expression is simpler; and if $N$ is a Fibonacci number, Theorem 5.1 of [1] gives the interpermit distances, and ( 4 ) reduces to a simple closed form expression for the mean waiting time. Unfortunately, it is not possible to bound the error analytically, though simulation results indicate that the results hold independently of $N$.

## REFERENCES


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