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Analysis and Comparison of Optimization Algorithm for Network Flow Control [1]

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Abstract

A modified Aitken-Extrapolation method is proposed for network flow control in this paper. Based on the dual model, this algorithm leads to a decentralized optimal solution to the problem of maximizing the aggregate source utility over the transmission rates. The contribution of this paper is to compare the proposed Aitken-extrapolation algorithm with some available optimization flow control algorithms in the recent literature and to analyze their rate of convergence. Simulation results demonstrate the consistency with our theoretical analysis.

1 Introduction

Effective rate control is required in order to control network flow and avoid congestion. A recent approach to flow control is based on optimization methods, e.g., [2~5]. In optimization-based flow control, each user is associated with a utility function, which suggests the portion of sharing of bandwidth with other users. A constraint is that the aggregate rate in one specific link should be within the link capacit [1]. The rate control objective is to achieve traffic rates that maximize the sum of the user utilities. A decentralized algorithm based on the dual model was proposed by S. Low and his coworkers in a nice paper [2] (also see [3]). They divided the primal problem into two sub-problems. One is an optimization problem of choosing the source rates to maximize the total benefit----subtract the total bandwidth cost from the total utilit function. This part is computed under a given price. On the contrary, the dua problem is to choose prices that can achieve the minimum of the dual objective function. This dual method has led the primal problem to a decentralized solution. A gradient projection method has been proposed in order to minimize the dual objective function. Two models are discussed: one i a synchronous distributed model; the other is an asynchronous distributed model. The gradient projection method was proved to converge under these two circumstances, but the speed of convergence is very slow. In [3], a Newton-like method was proposed which uses the diagona of Hessian matrix to form a scaled algorithm It has a faster convergence speed than the unscaled algorithm. But the authors of [2] and [3] have not given any theoretical analysis about the convergence speed of these algorithms. Motivated by this observation and the fact that the speed of convergence is very

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important in network flow control, we make a further step towards the problem of optimization-based flow control.

The main purpose of this paper is to introduce a modified Aitken-Extrapolation algorithm for flow control in telecommunications networks. Supported by theoretical analysis and computer simulations, it is shown that the propose Aitken-Extrapolation algorithm yields faster convergence than the previous gradient projection algorithm in [2] and improved transient performance than the Newton-like algorith in [3]. In particular, it is shown that the gradient projection algorithm is o first order and with geometric convergence if the step size is properly chosen. In addition, like the Newton-like algorith , the Aitken- extrapolation method, is superlinearly convergent.

The rest of the paper is presented as follows. Section 2 describes the (convex) optimization problem for a distributed network. Section 3 recalls the gradient projection method and a Newton-like algorithm proposed in the recent literature, and Section 4 presents our modified Aitken-Extrapolation algorithm. Our main theorems on the theoretical analysis and rate of convergence of these algorithms are stated in Section 5. Computer simulation shown in Section 6 validate the effectiveness of our modified Aitken-Extrapolation algorithm. Some brief concluding remarks are contained in Section 7.

The interested reader should contact us for an expanded version [8] of this conference paper for the omitted proofs.

2 The Optimization Problem

Optimization of communication networks has been studied in various contexts through the consideration of different objective functions and constraints. We will use a popular optimization problem model, which leads to the techniques in this paper

Suppose a network is composed of a set L={1,...,L} of unidirectional links of capacity $c_l, l \in \mathbf{L}$. A set of sources S={1,...,S} is sharing these links. For th sake of notational simplicity, introduce:

$L(s)$: a subset of **L**, which is a set of links which form the path for source s

U_s : a (positive, strictly concave, C^2) utility function for source s.

M_S: the maximum possible transmission rate for source s.

m_s : the minimum poss ble transmission rate for source s.

p_l : price of link l.

p : $[p_1, p_2, ..., p_L]^T$.

x_s : transmission rate of source s.

$I_s = [m_s, M_s]$: the range in which source rate x_s must lie.

$S(l) = \{s \in \mathbf{S} \mid l \in L(s)\}$: the set of sources that use link l.

Consider the maximization problem:

$$\text{P: } \max_{x_s \in I_s} \sum_s U_s(x_s) \qquad (1)$$

$$\text{s. t. } \sum_{s \in S(l)} x_s \le c_l, \qquad l = 1, ..., L \qquad (2)$$

The constraint (2) means the aggregate source rate at any link l should not exceed the capacity. Since the objective function $U_s(x_s)$ is strictly concave and hence, unique feasible optimal solution exists and must be a global solution.

The Lagrangian of the constrained problem is defined as

$$\begin{aligned} L(x,p) &= \sum_s U_s(x_s) - \sum_l p_l(\sum_{s\in S(l)} x_s - c_l) \\ &= \sum_s (U_s(x_s) - x_s \sum_{l\in L(s)} p_l) + \sum_l p_l c_l \end{aligned} \quad (3)$$

$U_s'(x_s)$ denotes the first derivative of U.

We can see it is hard to implemen distributed algorithms under the primal model. So a practical dual method is proposed in [2].

$$D(p) = \max_{x_s\in I_s} L(x,p) = \sum_s B_s(p^s) + \sum_l p_l c_l \quad (4)$$

where $B_s(p^s) = \max_{x_s\in I_s} U_s(x_s) - x_s p^s \quad (5)$

the dual problem is:

$$D:\ \min_{p\geq 0}\ D(p) \quad (6)$$

3 Gradient Projection Method

Based on this dual method, we will discuss some synchronous distributed algorithms. In [2], gradient projection algorithms are used to solve the synchronous problem. The link algorithm adjusts prices in opposite direction to the gradient $\nabla D(p)$:

$$p_l(t+1) = [p_l(t) - \gamma \nabla D(p(t))]^+ \quad (7)$$

we denote $[a]^+ = \max\{0, a\}$. $\nabla D(p)$ is a $L\times 1$ vector, the lth element of which is $\frac{\partial D}{\partial p_l}(p) = c_l - x^l(p)$, γ denotes the step size of the iteration. This algorithm is implemented in the REM algorithm discussed in [4]. But we can see from simulations that the convergence of gradient projection algorithm is slow. Instead of the gradient projection algorithm as used in [2][5], a faster convergent Newton-like algorithm is proposed in [3]. The source's algorithm is the same a the gradient projection algorithm; the link algorithm is revised as:

$$p_l(t+1) \approx [p_l(t) + \gamma H_{ll}^{-1}(x^l(t) - c_l)]^+$$

where $H_{ll} = \max\{\varepsilon, -\frac{x^l(t) - x^l(t-1)}{p_l(t) - p_l(t-1)}\}$,

and ε is a positive parameter used to mak $H = diag(H_{ll})$ positive definite. This parameter should be chosen carefully, becaus in case some diagonal elements in H are non-positive, ε is activated as the scaled elements. Here we would like to implement Aitken Extrapolation method for this dual model, which converge faster than gradient projection method and has no extra requirements for parameter selection.

4 Aitken Extrapolation [6]

The link algorithm can be thought of as a root finding process. Define $f_l(p) = c_l - x^l(p)$ and recall $p = [p_1, p_2, \ldots, p_L]^T$. Defining $\phi_l(p) = [p_l - \gamma f_l(p)]^+$, we can solve a fixed-point problem of the form $p_l = \phi_l(p)$ (or $p = \phi(p)$). Now we consider a sequence of p_l:

$$p_l(t)=\phi_l(p(t-1)),\bar{p}_l(t+1)=\phi_l(p(t))$$

$p_l(t)$ is the actual price of lin l at time t computed by the above gradient projection algorithm.

$\bar{p}_l(t+1)$ is an intermediate price of lin l at time $t+1$ computed by the gradient projection algorithm, but we will change it t $p_l(t+1)$ --- an actual price at time $t+1$.

Denote (x^*, p^*) as any pair of primal -dual optimal solution to the dual problem. As noticed in [2,3,7], p^* is not unique although x^* is unique becau e of the convexity assumptions made in the optimization problem. Then $p^* = \phi(p^*)$, $x^* = \phi(x^*)$. p_l^* and x_s^* are the lth and sth element of p^* and x^*, respectively.

From Mean Value Theorem or Taylor expansions, we have

$$p_l(t)-p_l^*=\phi_l(p(t-1))-\phi_l(p^*)=\phi_l^{'}(\theta_1)(p_l(t-1)-p_l^*)$$

$$\bar{p}_l(t+1)-p_l^*=\phi_l(p(t))-\phi_l(p^*)=\phi_l^{'}(\theta_2)(p_l(t)-p_l^*)$$

Whenever $p(+\infty)=p^*$ (we assume this gradient projection method is convergent), we can take

$\theta_1 = p(t_1) \quad t-1 \le t_1 \le +\infty$

and $\theta_2 = p(t_2) \quad t \le t_2 \le +\infty$

According to Aitken's formula, if p is near the optimal point p^*, $\phi_l^{'}(\theta_1)$ is almost equal to $\phi_l^{'}(\theta_2)$, i.e. $\phi_l^{'}(\theta_1) \approx \phi_l^{'}(\theta_2)$. So eliminate $\phi_l^{'}(\theta_1), \phi_l^{'}(\theta_2)$ from the above two equations to arrive at:

$$\frac{p_l(t)-p_l^*}{\bar{p}_l(t+1)-p_l^*} \approx \frac{p_l(t-1)-p_l^*}{p_l(t)-p_l^*} \tag{8}$$

Then we obtai

$$p_l^* \approx \bar{p}_l(t+1)-\frac{(\bar{p}_l(t+1)-p_l(t))^2}{\bar{p}_l(t+1)-2p_l(t)+p_l(t-1)} \tag{9}$$

5 Speed of Convergence

The convergence of the Aitke -Extrapolation algorithm in scalar form is proved i [6]. Here we deal with the case where the mapping function ϕ is vecto -valued, leading to som technical complication.

Theorem 1: *The gradient projection algorith is convergent with linear convergence speed if the stepsize is appro riately chosen.*

Theorem 2: *The Newton-like algorithm is superlinearly convergent if* $\gamma = 1$. *It is approximately linearly convergent if* $0 < \gamma < 1$ and diverges if $\gamma >> 1$.

Theorem 3: *The Aitken-Extrapolation algorithm proposed in Section III.B is similar with Newton-like algorithm, so the convergence rate is also superlinea .*

The proofs of these main results are given in the full versio [8] of our pape .

Remark 1 As said previously, the gradient projection algorithm was first proposed in [2] for network flow control but the authors did not analyze the rate of convergence. Theorem 1 complements the theoretical result of [2] by providing explicit sufficient conditions for geometric convergence.

6 Simulation Results

We have based our simulation on the same model as in [3]. In Fig 1, five connections $S_i - D_i$ (i=1,2...,4) with source S_i and destinatio D_i share four links. Connectio $S_1 - D_1$ spanned links 1,2,3,4; $S_2 - D_2$ spanned link 1; $S_3 - D_3$ spanned link 2; $S_4 - D_4$ spanned link 3; $S_5 - D_5$ spanne link 4. All links were identical with capacity equal to 220 packets per second easuring interval. Source S_1 transmitted data fro time 0s to time 300s. The start times of the other sources are staggered with 40s interval. Sources S_2, S_3, S_4 remain active until 240s, and S_5 turned off earliest at time 200s. The utility functions of the sources were set to $\omega_s \log(1 + x_s£$, with ω_s equal to 4×10^4 for source S_1 and 1×10^4 for sourc S_2, S_3, S_4, S_5 respectively. The target bandwidth (c_l) was set at 200 packets per 1s measuring interval, which gives zero equilibrium buffer occupancy.

The initial conditions of this algorithm are set as follows:

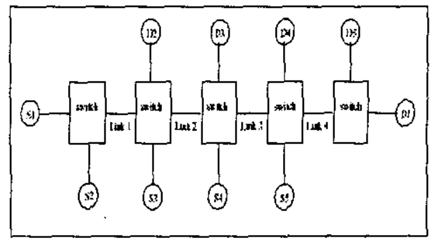

Fig.1 Network Topology [3]

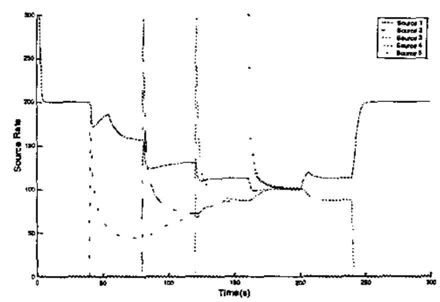

Fig.2 Source rates under gradient projection algorithm

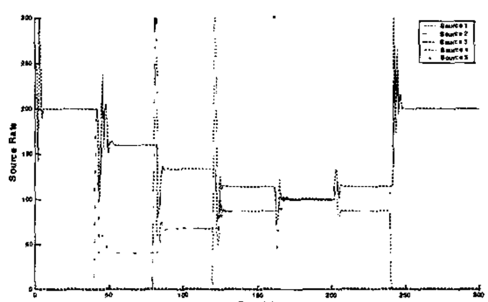

Fig.3 Source rates under Aitken-Extrapolation algorithm

$\gamma = 0.15$ for gradient projection algorithm in order to achieve convergence; $\gamma = 1$ for Aitken-Extrapolation algorithm.

$$x_{s \in S} = 0 \quad s = 1, \ldots, S$$

$$p_{l \in L} = 0 \quad l = 1, \ldots, L$$

$$m_s = 20\, packets/\sec$$

$$M_s = 300\, packets/\sec$$

The simulation results have shown that Aitke -Extrapolation algorithm achieves faster speed of convergence. In addition, the magnitude of the rate fluctuation under the Aitke -Extrapolatio algorithm is much smaller than that with the gradient projection method.

7 Conclusion

The faster convergence rate implies less overloading and hence much less buffer requirement at the links [3]. In this paper, we have examined the application of a practical Aitke -Extrapolation algorithm in network flow control. Its superiority over gradient projection algorithms is illustrated by theoretical analysis and simulations. For the proposed Aitken-Extrapolation algorithm, w use three values at time t -1, t, t+1 to extrapolate a new value that is closer to the optimal one. Employing this methodology, it is not difficult to get a more precise algorithm based on the extrapolation of more than three points. Of course more memory is needed to restore the old values (prices and sources rates) and more computation task is required. Another advantage of the proposed algorith is that, *even if the original projection algorithm is not convergent but of the first order, the Aitken-Extrapolation algorithm is still convergent* We are currently investigating *asynchronous* distributed algorithms with this method, and will report on new findings separately.

8 References

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