Analysis of Generalized Processor Sharing Systems Using Matrix Analytic Methods

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Abstract — **Generalized Processor Sharing (GPS) is an important scheduling discipline because it enables bandwidth sharing with work conservation and traffic isolation properties. Although Markov Modulated Fluid Processes (MMFP) captures the fine dynamics of the sources and is expected to give tight performance bounds, the analysis of MMFP sources with a GPS server is usually difficult because of the large state space and the coupled services of the classes. Matrix analytic methods [8], which yield great numerical accuracy and stability, are effective alternatives to the spectral analysis approach (e.g. [12]). In this paper, we apply Matrix Analytic methods for fluid flows as introduced by Ramaswami [11] to the analysis of GPS systems fed by MMFP sources. We propose a new technique to calculate the tail distributions of the classes where matrices processed are of smaller sizes, which greatly reduces the computation complexity. Numerical results illustrate the efficiency and accuracy of the technique. We also investigate the *Caudal Characteristics* of GPS queues which further illustrate the effectiveness of the GPS scheduling discipline.**

I. Introduction

The main objective for next generation networks is accommodating a variety of services with different traffic characterizations and Quality of Service (QoS) requirements. Scheduling is used in switches and routers to enforce service differentiation. Among the scheduling disciplines, Generalized Processor Sharing (GPS), along with its variants such as Weighted Round Robin, has such desirable properties as bandwidth sharing with traffic isolation, fairness provisioning, and service differentiation, making it suitable for QoS provisioning.

GPS has been widely studied [1, 2, 3, 4]. Most previous work takes bounding approaches and focuses on general arrival processes, with deterministic or stochastic settings. GPS systems are studied with various source characterizations, such as Poisson with symmetric service sharing [1], leaky bucket regulated sources [2], exponentially bounded burstiness sources [3], and long-tailed sources [4]. By the notion of *feasible ordering/partitioning*, a GPS system can be reduced to a priority system and hence performance bounds are readily obtained [2, 3]. A queue decomposition technique is proposed in [3] to de-couple the service. These results are generally expected to be loose since the finer dynamics of the sources are not exploited [5].

Matrix analytic methods, introduced by M. F. Neuts [8] for the treatment of certain special two-dimensional Markov chains, take advantage of the structural simplicity of the embedded Markov chains and provide numerically stable approaches for the treatment of such systems. There is a rich literature on matrix analytic methods, and interested readers can refer to [9] for a review of the recent developments. Sengupta [10] and Ramaswami [11] showed that under certain assumptions, the steady state probability distribution of a fluid queue has a matrix-exponential form which is a continuous analog of matrix geometric methods. Ramaswami derived the technique of analyzing the classical Markovian fluid queueing model [12] using matrix analytic methods [11]. By appealing to the skip-free nature of fluid queues and time reversibility theory, the computation of the steady state distribution of a fluid queue fed by Markov Modulated Fluid Processes (MMFP) is reduced to the analysis of a discrete time, discrete state space quasi-birth-death (QBD) model. It is shown in [11] that this approach yields great numerical accuracy and stability.

In this paper we study a multiple-class GPS system with classes modeled as MMFP. Markovian processes have been widely used to model network traffic and are efficient in modeling Voice over IP traffic [7, 13]. It is well known that data traffic is *self-similar*, which poses a great challenge to network control and QoS provisioning. Previous work show that *Long Range Dependent* traffic,

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such as VBR video, can be adequately approximated by *Short Range Dependent* traffic models for traffic engineering purposes [14, 15]. Furthermore, recent evidence shows that aggregating web traffic causes it to smooth out as rapidly as Poisson traffic [16]. These make it possible to investigate certain aspects of the impact on the performance of the long-range correlation structure within the confines of traditional Markovian analysis [15].

We introduce an efficient technique for deriving the tail distribution of a MMFP class in a multiple-class, infinite buffer GPS system. In our previous work [6], we proposed a tight service bound for a tagged queue in the system. Applying this service bound, we first transform the GPS system into a simpler deterministic service queue for the tagged class; then matrix analytic methods is applied to get the tail distribution for the tagged class. Compared with spectral analysis, this method processes smaller matrices of the order of the number of overload states or underload states. There is also no need to solve the linear system of boundary conditions for coefficients. Lower computation effort allows us to handle larger systems. Also this method is less liable to the numerical problems incurred in the classical spectral analysis [12] and yields greater numerical accuracy [11]. Since the state space increases exponentially with the number of sources, this method is more effective for the QoS analysis of links carrying a moderate number of sources, such as access links or Virtual Private Networks (VPN). For larger systems, effective bandwidth based algorithms, e.g. [7, 19], are recommended since they are relatively more scalable.

The rest of the paper is organized as follows: in Section II the system model and the service bound are presented. In Section III we review the analysis of stochastic fluid flow models with matrix analytic methods in [11] and presents several useful extensions. Numerical results are given in Section V. Section VI is the conclusions.

II. The GPS System with MMFP Sources

A. The System Model

GPS is a work conserving scheduling discipline in which N traffic classes share a deterministic server with rate c [3]. There is a set of parameters ω_i, $i = 1, \ldots, N$, called GPS *weights*. Each class is guaranteed a service rate $g_i = \omega_i c$, and the residual service of the non-backlogged classes is distributed to the backlogged classes in proportion to their weights.

The system is shown in Fig.1. Each class i, with instant rate $r_i(t)$, is modeled as a MMFP with state space $\mathcal{S}_i$, rate matrix $\mathbf{R}_i$, and infinitesimal generator $\mathbf{T}_i$. $\mathbf{T}_i$ governs the transitions between the states and $r_i(t) = \mathbf{R}_i(\tau_i, \tau_i)$ when class i is in state τ_i at time t. Assume $\sum_i \overline{\lambda}_i \leq c$, which guarantees the ergodicity of the system; and $\overline{\lambda}_i < g_i$, $i = 1, \ldots, N$, where $\overline{\lambda}_i$ is the average rate of class i. The buffer is infinite. Each class has its own logical queue with occupancy $X_i(t)$.

We are interested in the tail distribution of class i's queue occupancy, which upper bounds the loss class i experiences in a finite buffer system and can also be used to bound its delay distribution.

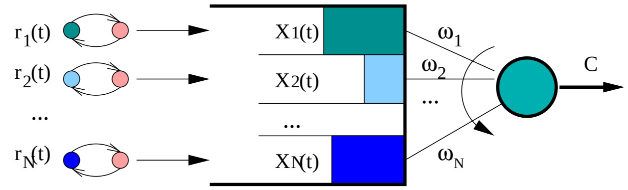

Fig. 1: The system model

B. A Service Bound

In our previous work [6], we proposed a lower bound for the service that a class receives in the GPS system (LMP bound), which decouples the correlated GPS service as shown in Fig.2.

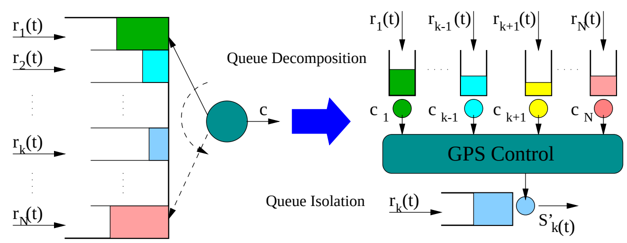

Fig. 2: The queue decomposition technique.

It takes two steps to obtain the LMP bound for the tagged class i. First, the departure process of each class, $r_j^{'}(t)$, $j \neq i$, is approximated by assuming the service rate is g_j. The generator matrix of $r_j^{'}(t)$ is $\mathbf{T}_j^{'}$ and the state space of the departure process, $\mathcal{S}_j^{'}$, consists of several underload states and one overload state whose rate equals to g_j if g_j is less than the peak rate of $r_j(t)$; otherwise the departure process is identical to $r_j(t)$. The technique in [17] is used to characterize the departure processes. Secondly, class i's service rate, $s_i^{'}(t)$, is the guaranteed service rate g_i increased by its share of the residual service rate from all un-backlogged queues according to the GPS weights of all backlogged classes as shown below.

$$s_i^{'}(t) \equiv g_i + \frac{\omega_i}{\omega_i + \sum_{k \in B(t)} \omega_k} \sum_{j \in \overline{B}(t)} (g_j - r_j^{'}(t)). \qquad (1)$$

$B(t)$ in (1) is the set of the backlogged queues, and $\overline{B}(t)$ is the set of un-backlogged queues at time t. Logical queue i's occupancy distribution can be derived from a queue with input $r_i(t)$ and modulated service $s_i^{'}(t)$ (both are MMFPs).

It is proved in [6] that the LMP bound is an approximate lower bound of the service class i receives, hence the tail distribution obtained from (1) is an approximate upper bound of class i's tail distribution. The tightness of the LMP bound depends on how accurately the departure processes are modeled. In [17], it is argued that the output characterization of a FIFO queue with MMFP input is accurate when the non-empty buffer probability is lower than 10^{-3}, which is not atypical for real-time traffic with a stringent delay requirement. For such systems, we believe the LMP bound can be reasonably tight for the cases of interest.

C. Transform To A Deterministic Service Queue

In the previous section, we decomposed the GPS system into a queue with modulated service. Next we make a simple transform to change it into a deterministic service queue.

The service bound in (1) can be expressed as:

$$s_i^{'}(t) \quad = \quad c_i^* - r_i^*(t) \tag{2}$$

where $r_i^*(t) = \frac{c\sum_{k\in B(t)}\omega_k + \omega_i \sum_{j\neq i} r_j^{'}(t)}{\omega_i + \sum_{k\in B(t)}\omega_k}$, and $c_i^* = g_i + c$. Then the drift of logical queue i is:

$$\frac{d}{dt}X_i(t) \quad = \quad r_i(t) + r_i^*(t) - c_i^*$$

This is identical to the system equation of a deterministic service queue with service rate c_i^*, and two MMFP sources $r_i(t)$ and $r_i^*(t)$, as illustrated in Fig.3. The second source has rate $r_i^*(t)$ and generator matrix $\mathbf{T}_i^* = \mathbf{T}_1^{'} \oplus \ldots \oplus \mathbf{T}_k^{'} \oplus \ldots \oplus \mathbf{T}_N^{'}, k \neq i$, where $\oplus$ is the Kronecker sum operator.

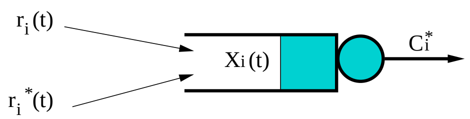

Fig. 3: The equivalent model

III. Matrix Analytic Methods for Stochastic Fluid Flows

In [11], Ramaswami applied matrix analytic methods to fluid flow analysis. By appealing to the skip-free nature of fluid level fluctuations and time reversibility theory, the computation of the steady state distribution of a Markov fluid FIFO queue is reduced to the analysis of a discrete time, discrete state space QBD model. Such QBD models are well studied and many computational algorithms are available in the literature for them. A very good reference is [9]. In this section, we first review the main results in [11], and then we derive three useful corrollaries.

A. Fluid Flow Analysis

Suppose the fluid flow source is governed by a continuous time, irreducible Markov process with state space $\{1, \ldots, m, m+1, \ldots, m+n\}$ and infinitesimal generator $\mathbf{T}$. The net rate of input to the infinite buffer is assumed to be $d_i > 0$ when the Markov chain is in state i, $i \leq m$, and $d_j < 0$ when the system is in state j, $j > m$.

Let $\mathbf{\Pi} = [\mathbf{\Pi}_1, \mathbf{\Pi}_2]$ be the steady state probability vector of the process, while $\mathbf{\Pi}_1$ is of order $m \times 1$ and $\mathbf{\Pi}_2$ of order $n \times 1$. Define $\mathbf{\Delta} = diag(\mathbf{\Pi})$, $\tilde{\mathbf{T}} = \mathbf{\Delta}^{-1}\mathbf{T}'\mathbf{\Delta}$, and $\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{bmatrix}$, where $\mathbf{D}_1 = diag(d_1, d_2, \ldots, d_m)$ and $\mathbf{D}_2 = diag(d_{m+1}, d_{m+2}, \ldots, d_{m+n})$. Also let $\tilde{\mathbf{S}} = \mathbf{D}^{-1}\tilde{\mathbf{T}}$ and partition $\tilde{\mathbf{S}}$ into $\tilde{\mathbf{S}} = \begin{bmatrix} \tilde{\mathbf{S}}_{11} & \tilde{\mathbf{S}}_{12} \\ \tilde{\mathbf{S}}_{21} & \tilde{\mathbf{S}}_{22} \end{bmatrix}$, where $\tilde{\mathbf{S}}_{11}$ is of order $m \times m$.

Theorem 1 [11]: The stationary distribution of the fluid flow is *phase type* with representation $PH(\mathbf{\Lambda}, \mathbf{U})$ of order m, where $\mathbf{\Lambda} = \mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{W}$. The tail probability of the queue distribution is given by:

$$G(x) = \mathbf{\Lambda} e^{\mathbf{U}x}\mathbf{1}, \qquad \text{for } x \geq 0. \tag{3}$$

where

$$\mathbf{U} = \tilde{\mathbf{S}}_{11} + \tilde{\mathbf{S}}_{12}\int_0^\infty e^{\tilde{\mathbf{S}}_{22}y}\tilde{\mathbf{S}}_{21}e^{\mathbf{U}y}dy \tag{4}$$

and

$$\mathbf{W} = \int_0^\infty e^{\tilde{\mathbf{S}}_{22}}\tilde{\mathbf{S}}_{21}e^{\mathbf{U}y}dy. \tag{5}$$

Choose a number $\theta \geq max_i(-\tilde{\mathbf{S}}_{ii})$, and let $\mathbf{M}_1 = \theta^{-1}\mathbf{U} + \mathbf{I}$, $\tilde{\mathbf{P}}_{ii} = \theta^{-1}\tilde{\mathbf{S}}_{ii} + \mathbf{I}$, and $\tilde{\mathbf{P}}_{ij} = \theta^{-1}\tilde{\mathbf{S}}_{ij}$, for $i \neq j$. Define matrices

$$\mathbf{A}_2 = \begin{bmatrix} \tilde{\mathbf{P}}_{11} & \mathbf{0} \\ \frac{1}{2}\tilde{\mathbf{P}}_{21} & \mathbf{0} \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{P}}_{12} \\ \mathbf{0} & \frac{1}{2}\tilde{\mathbf{P}}_{22} \end{bmatrix}, \mathbf{A}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{I} \end{bmatrix}. \tag{6}$$

Theorem 2 [11]: The rate matrix of the QBD defined by (6) is given by:

$$\mathbf{G} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{W} & \mathbf{0} \end{bmatrix}. \tag{7}$$

B. Computing The Rate Matrix

The G-matrix of the QBD defined by (6) is the solution of a non-linear matrix equation, and can be found by successive substitutions using (8) [8, 9].

$$\mathbf{G}_n = (\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_0\mathbf{G}_{n-1})^{-1}\mathbf{A}_2 \tag{8}$$

Note the matrices in (6) are quite sparse. Further exploiting this fact and the special structure of the $\mathbf{G}$ matrix, a new iteration scheme of computing $\mathbf{M}_1$ and $\mathbf{W}$ can be designed as follows:

$\mathbf{W} = \frac{1}{2}(\mathbf{I} - \frac{1}{2}\tilde{\mathbf{P}}_{22})^{-1}\tilde{\mathbf{P}}_{21}$;
do
  $\mathbf{W}_{old} = \mathbf{W}$;
  $\mathbf{W} = \frac{1}{2}(\mathbf{I} - \frac{1}{2}\tilde{\mathbf{P}}_{22} - \frac{1}{2}\mathbf{W}\tilde{\mathbf{P}}_{12})^{-1}(\mathbf{W}\tilde{\mathbf{P}}_{11} + \tilde{\mathbf{P}}_{21})$;
until $\| \mathbf{W} - \mathbf{W}_{old} \|_\infty < \varepsilon$;
$\mathbf{M}_1 = \tilde{\mathbf{P}}_{11} + \tilde{\mathbf{P}}_{12}\mathbf{W}$;
output $\mathbf{M}_1$ and $\mathbf{W}$.

This iteration scheme has the following convergence property:

Corollary 1 : The iterative scheme proposed above converges linearly.

Proof: Plug in (6) and (7) into (8), we have

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{W} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\tilde{\mathbf{P}}_{12} \\ -\frac{1}{2}\mathbf{W} & \mathbf{I}-\frac{1}{2}\tilde{\mathbf{P}}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{P}}_{11} & \mathbf{0} \\ \frac{1}{2}\tilde{\mathbf{P}}_{21} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{\mathbf{P}}_{11}+\tilde{\mathbf{P}}_{12}\mathbf{W} & \mathbf{0} \\ \frac{1}{2}(\mathbf{I}-\frac{1}{2}\tilde{\mathbf{P}}_{22}-\frac{1}{2}\mathbf{W}\tilde{\mathbf{P}}_{12})^{-1}(\mathbf{W}\tilde{\mathbf{P}}_{11}+\tilde{\mathbf{P}}_{21}) & \mathbf{0} \end{bmatrix},$$

which gives:

$$\begin{cases} \mathbf{M}_1 = \tilde{\mathbf{P}}_{11}+\tilde{\mathbf{P}}_{12}\mathbf{W} \\ \mathbf{W} = \frac{1}{2}(\mathbf{I}-\frac{1}{2}\tilde{\mathbf{P}}_{22}-\frac{1}{2}\mathbf{W}\tilde{\mathbf{P}}_{12})^{-1}(\mathbf{W}\tilde{\mathbf{P}}_{11}+\tilde{\mathbf{P}}_{21}) \end{cases}$$

Thus this algorithm is the same as the algorithm based on (8) in [9] and they have the same linear convergence characteristics. ■

The matrices used in this scheme are of the order of m or n. Recall that m is the number of overload states and n is the number of underload states. We process smaller matrices here rather than directly calculating $\mathbf{G}$ from the $\mathbf{A}_i$'s. Our experiments show that, for typical cases, this scheme requires the same number of iterations as the linear convergence algorithm in [9] and is about 10 times faster.

C. Caudal Characteristics of Fluid Queues

It is well known that if a QBD is positive recurrent then its steady-state probability vector $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1, \mathbf{\Pi}_2, \ldots\}$ has a matrix geometric form and decays geometrically with rate η. Thus η essentially describes the tail behavior of the model, and is called the *Caudal Characteristics factor* of the QBD [18]. It is closely related to the asymptotic decay rate, or the *effective bandwidth* in large deviation theory [19]. Here we define the Caudal Characteristics factor for the fluid queues.

Corollary 2 : The *Caudal Characteristics factor* of the fluid queue defined in Section III-A, η, is

$$\begin{aligned} \eta &= \mathrm{MRE}(\mathbf{U}) = \theta(\mathrm{MRE}(\mathbf{M}_1)-1) \\ &= \theta(\mathrm{MRE}(\mathbf{G})-1), \end{aligned} \tag{9}$$

where $\mathrm{MRE}(\mathbf{X})$ calculates the maximum real eigenvalue of matrix $\mathbf{X}$.

Proof: Assume ψ_i and ξ_i be the normalized left and right eigenvectors of $\mathbf{U}$ with corresponding eigenvalues σ_i, $i = 1, \ldots, m$. Then

$$\begin{aligned} G(x) &= \mathbf{\Lambda} \begin{bmatrix} \xi_1' \\ \ldots \\ \xi_m' \end{bmatrix}' \begin{bmatrix} e^{\sigma_1 x} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & e^{\sigma_m x} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \ldots \\ \psi_m \end{bmatrix} \mathbf{1} \\ &= \sum_{i=1}^{m} < \mathbf{\Lambda}, \xi_i > e^{\sigma_i x} < \psi_i, \mathbf{1} > . \end{aligned}$$

Suppose $\eta = \max_i\{\sigma_i\}$ and η is the k_{th} eigenvalue of $\mathbf{U}$. When x gets large, the η term dominates.

$$G(x) \simeq < \mathbf{\Lambda}, \xi_k > e^{\eta x} < \psi_k, \mathbf{1} > .$$

Tab. 1: On-off Source parameters

-	α	β	λ
Type 1	0.40	1.00	1.00
Type 2	0.40	1.00	1.20
Type 3	1.00	1.00	0.61
Type 4	0.56	0.83	8.0

Thus η determins the tail behavior of the fluid queue for large buffers. From (7), it is obvious that $\mathbf{M}_1$ has the same eigenvalues as $\mathbf{G}$. From the definition of $\mathbf{M}_1$, the relationship between its eigenvalues and that of $\mathbf{U}$ can be derived. ■

For the discrete-time, discrete space QBD with rate matrix $\mathbf{G}$, algorithms are proposed in [18] to compute its *Caudal Characteristic factor*, which is more interesting when the order of $\mathbf{G}$ is so large that an exact computation of $\mathbf{G}$, and therefore also of the exact tail distribution is not feasible. If the phase space is decomposable, fast algorithms are provided in [18].

Corollary 3 :Assume η is the k_{th} eigenvalue of $\mathbf{U}$. let ψ_k and ξ_k denote the k_{th} left and right eigenvectors of $\mathbf{U}$. When x gets large, the tail distribution of the fluid queue can be approximated by:

$$G(x) \simeq < \mathbf{\Lambda}, \xi_k >< \psi_k, \mathbf{1} > e^{\eta x}, \tag{10}$$

There are efficient methods to calculate the dominant eigenpair of a matrix available in the matrix theory literature. Eq.(10) can be used for the cases where a fast approximation is needed.

IV. Numerical Investigations

In this section we present some of the numerical results, which illustrate the quantitative and qualitative aspects of the technique.

In some of the experiments we have done, we use classes of sources where each class is the aggregation of a number of on-off sources. These bursty sources are used in modeling packet voice for traffic engineering purposes [6, 13]. The source parameters are given in Table 1, where α and β are the transition rates from off to on, and from on to off, respectively; λ is the rate when the source is on. Although the classes considered here consist of homogeneous sources, the technique also applies to the cases where a class consisting of heterogeneous sources. The analysis results are compared with fluid simulation with 95% confidence intervals to verify their correctness.

Consider a 3-class GPS system where Class 1 has 10 Type 1 sources; class 2 has 10 Type 2 sources; and class 3 has 10 Type 3 sources. The service rate is $c = 15.1$, while the GPS weights are $\omega_1 = 0.33$, $\omega_2 = 0.40$, and $\omega_3 = 0.27$, respectively. The system load is about 61%, which is not atypical in traffic engineering. Fig.4 shows the tail distributions of the classes. It can be seen that

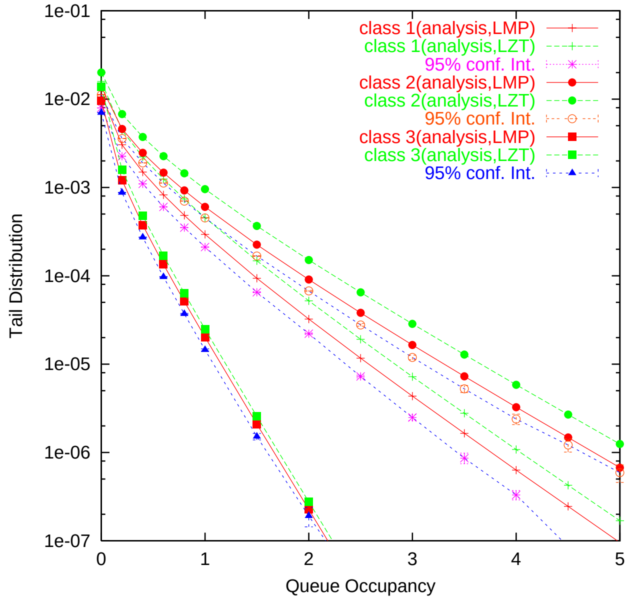

Fig. 4: Tail distributions of a 3-queue GPS system

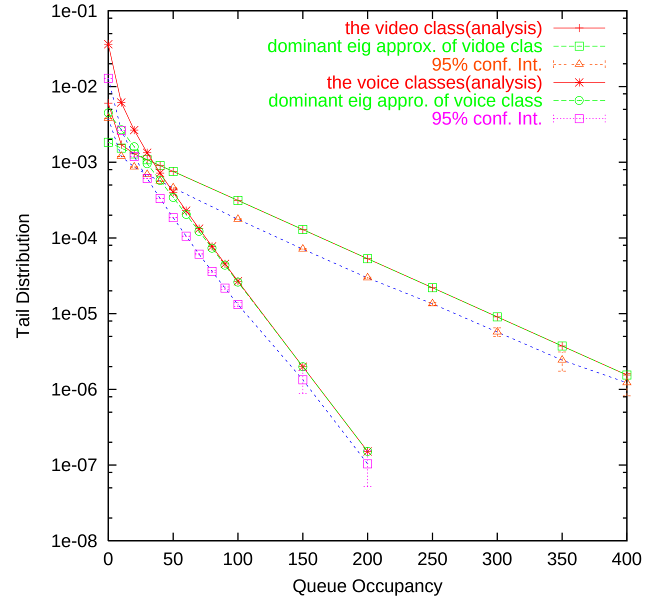

Fig. 5: Tail distributions of a 3-queue GPS system, where class 1 has two video sources and class 2 and 3 have 20 voice sources each

under this load our service bound (1) gives a good approximation. All three analysis curves match simulations for the whole buffer region. We also plot the analytic results for the tail distributions of the classes using the service bound in [5] (LZT bound) for comparison purpose. Like the LMP bound, LZT bound also requires the output characterization of all non-tagged classes and the computing of all the eigenvalues and eigenvectors of the derived system. The computation complexity of the LMP bound and LZT bound are comparable. The LMP bound results match the simulation more closely than that of the LZT bound. As an example on the reduced matrix sizes using this approach, for class 1, the derived equivalent system, Fig.3, has 616 states, while the $\mathbf{U}$ matrix of this system is of size 100×100.

To further illustrate the effectiveness of the technique, we study a 3-queue GPS system with a video class and two voice classes. The video model is from [14], in which a four-state Discrete-time Markov Modulated Poisson Process (DMMPP) is used to model video traffic. A video source has transition matrix $\mathbf{T}$ and rate vector $\mathbf{R}$ as given below. Note $\widehat{\alpha_i} = (1-\alpha_i)$ and $\widehat{\beta_i} = (1-\beta_i)$ in $\mathbf{T}$. The parameters are matched from the MPEG-1 *Star Wars* movie in [14], and are given in Table 2. We use its fluid equivalent for the analysis and simulations. A voice source is modeled using parameters of Type 4 in Table 1. Class 1 consists of 2 video sources, while class 2 and class 3 are identical, consisting of 20 voice sources each. The GPS weights for the classes are 0.54, 0.23, and 0.23, respectively, and the service rate is 306.6. Fig.5 plots the tail distributions of the classes. Again, the analysis results match the simulation results closely for the whole buffer region. We also plot the tails of the classes using (10). The approximations almost overlap with the exact analysis except for small buffer regions. The equivalent system derived for class 1 has 1600 states, while the $\mathbf{U}$ matrix obtained is of size of 14×14.

Tab. 2: Video Source parameters

α_1	α_2	β_1	β_2
0.0018	0.00064	0.1568	0.0234
λ_1	λ_2	λ_3	λ_4
95.24	58.9	73.92	37.58

$$\mathbf{T} = \begin{bmatrix} \widehat{\alpha_1}\widehat{\alpha_2} & \widehat{\alpha_1}\alpha_2 & \alpha_1\widehat{\alpha_2} & \alpha_1\alpha_2 \\ \widehat{\alpha_1}\beta_2 & \widehat{\alpha_1}\widehat{\beta_2} & \alpha_1\beta_2 & \alpha_1\widehat{\beta_2} \\ \beta_1\widehat{\alpha_2} & \beta_1\alpha_2 & \widehat{\beta_1}\widehat{\alpha_2} & \widehat{\beta_1}\alpha_2 \\ \beta_1\beta_2 & \beta_1\widehat{\beta_2} & \widehat{\beta_1}\beta_2 & \widehat{\beta_1}\widehat{\beta_2} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}$$

Next we examine the *Caudal Characteristics* (η defined in (9)) of the classes with a GPS server. In Fig.6, we plot the Caudal Characteristic curve of class 1 in a 3-class GPS system for four different GPS weights. In the 3-class GPS system, class 1 has two Type 1 sources, class 2 has two Type 2 sources, and class 3 has two Type 3 sources. In Fig.6, ω_1 increases from 1/5 to 1/2, while $\omega_2 = \omega_3 = \frac{1}{2}(1-\omega_1)$. The system load is varied from 0.35 to 1 by adjusting the service rate. Different GPS weights yield different Caudal Characteristic curves. The higher the GPS weight, the lower the η. This shows the separation and protection features of GPS servers and allows for the selection of the right GPS weights to meet QoS requirements.

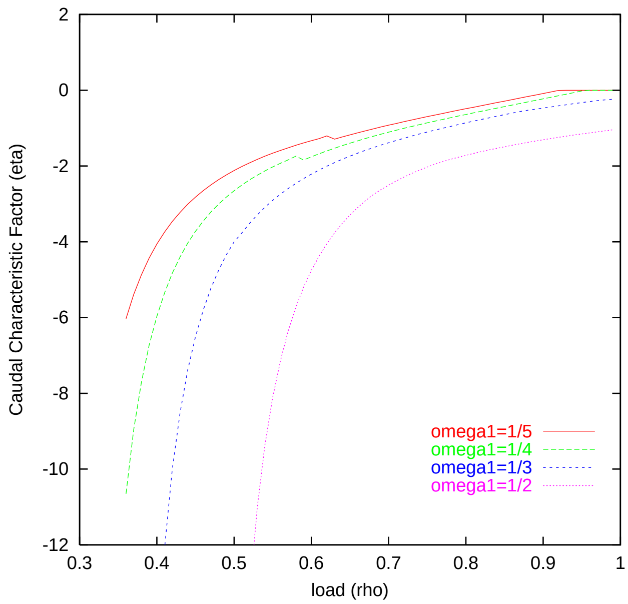

Fig. 6: Caudal Characteristics of class 1 queue in a three-queue GPS system versus system load and with different GPS weights

V. Conclusions

In this paper we present a simple and efficient analytical technique for determining the tail distributions of MMFP sources in a GPS system. The GPS system is decomposed into a FIFO queue fed by two MMFP sources for the tagged class using the LMP bound. By applying matrix analytic methods to the fluid flow, the tail distribution of the tagged class is obtained. This technique requires less computational effort than a spectral analysis because smaller matrices are processed. Also this approach has the well-known numerical stability advantages of matrix analytic methods. It is therefore an effective alternative to the spectral analysis approach. Numerical results illustrate the accuracy and efficiency of the technique.

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